6-8 Roots and Zeros

Main Ideas

- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

GET READY for the Lesson

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time.

Suppose the equation $M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

Types of Roots You have already learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

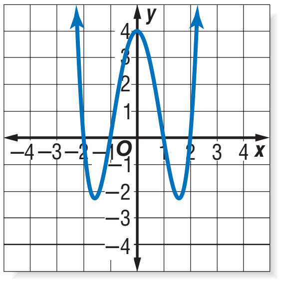
KEY CONCEPT *Zeros, Factors, and Roots*

Let $f(x) = a_n x^n + \ldots + a_1x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of the polynomial function $f(x)$.
- $x - c$ is a factor of the polynomial $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.

In addition, if c is a real number, then $(c, 0)$ is an intercept of the graph of $f(x)$.

The graph of $f(x) = x^4 - 5x^2 + 4$ is shown at the right. The zeros of the function are -2, -1, 1, and 2. The factors of the polynomial are $x + 2$, $x + 1$, $x - 1$, and $x - 2$. The solutions of the equation $f(x) = 0$ are -2, -1, 1, and 2. The x-intercepts of the graph of $f(x)$ are $(-2, 0)$, $(-1, 0)$, $(1, 0)$, and $(2, 0)$.



Study Tip

Look Back

For review of **complex numbers,** see Lesson 5-4.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra.**

KEY CONCEPT *Fundamental Theorem of Algebra*

Every polynomial equation with complex coordinates and degree greater than zero has at least one root in the set of complex numbers.

EXAMPLE 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. $x^2 - 8x + 16 = 0$

$x^2 - 8x + 16 = 0$ Original equation

$(x - 4)^2 = 0$ Factor the left side as a perfect square trinomial.

$x = 4$ Solve for x using the Square Root Property.

Since $x - 4$ is twice a factor of $x^2 - 8x + 16$, 4 is a double root. So this equation has one real repeated root, 4.

b. $x^4 - 1 = 0$

$$x^4 - 1 = 0$$

$$(x^2 + 1)(x^2 - 1) = 0$$

$$(x^2 + 1)(x + 1)(x - 1) = 0$$

$$x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x^2 = -1 \qquad x = -1 \qquad x = 1$$

$$x = \pm\sqrt{-1} \text{ or } \pm i$$

This equation has two real roots, 1 and −1, and two imaginary roots, i and $-i$.

CHECK Your Progress

1A. $x^3 + 2x = 0$

1B. $x^4 - 16 = 0$

Reading Math

Roots In addition to double roots, equations can have triple or quadruple roots. In general, these roots are referred to as *repeated roots*.

Real-World Link

René Descartes (1596–1650) was a French mathematician and philosopher. One of his best-known quotations comes from his *Discourse on Method*: "I think, therefore I am."

Source: *A History of Mathematics*

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

KEY CONCEPT — *Corollary*

A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.

Similarly, a polynomial function of nth degree has exactly n zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

KEY CONCEPT — *Descartes' Rule of Signs*

If $P(x)$ is a polynomial with real coefficients, the terms of which are arranged in descending powers of the variable,

- the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

EXAMPLE 2 Find Numbers of Positive and Negative Zeros

2 State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$.

Since $p(x)$ has degree 5, it has five zeros. However, some of them may be imaginary. Use Descartes' Rule of Signs to determine the number and type of real zeros. Count the number of changes in sign for the coefficients of $p(x)$.

$$p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$$

yes	no	yes	yes	yes
+ to −	− to −	− to +	+ to −	− to +

Since there are 4 sign changes, there are 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$p(x) = (-x)^5 - 6(-x)^4 - 3(-x)^3 + 7(-x)^2 - 8(-x) + 1$$
$$= -x^5 - 6x^4 + 3x^3 + 7x^2 + 8x + 1$$

no	yes	no	no	no
− to −	− to +	+ to +	+ to +	+ to +

Since there is 1 sign change, there is exactly 1 negative real zero.
Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero. Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	1	0	$4 + 1 + 0 = 5$
2	1	2	$2 + 1 + 2 = 5$
0	1	4	$0 + 1 + 4 = 5$

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Study Tip

Zero at the Origin
Recall that the number 0 has no sign. Therefore, if 0 is a zero of a function, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

CHECK Your Progress

2. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $h(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$.

Find Zeros We can find all of the zeros of a function using some of the strategies you have already learned.

EXAMPLE 3 Use Synthetic Substitution to Find Zeros

3 Find all of the zeros of $f(x) = x^3 - 4x^2 + 6x - 4$.

Since $f(x)$ has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = x^3 - 4x^2 + 6x - 4 \qquad f(-x) = -x^3 - 4x^2 - 6x - 4$$

$f(x)$: yes yes yes

$f(-x)$: no no no

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of $f(-x)$, $f(x)$ has no negative real zeros. Thus, $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and $f(0)$ is -4, begin by evaluating $f(x)$ for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find $f(a)$ for several values of a.

x	1	-4	6	-4
1	1	-3	3	-1
2	1	-2	2	0
3	1	-1	3	5
4	1	0	6	20

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

Study Tip

Finding Zeros

While direct substitution could be used to find each real zero of a polynomial, using synthetic substitution provides you with a depressed polynomial that can be used to find any imaginary zeros.

From the table, we can see that one zero occurs at $x = 2$. Since the depressed polynomial of this zero, $x^2 - 2x + 2$, is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, $x^2 - 2x + 2 = 0$.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Quadratic Formula

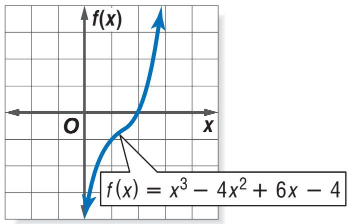
$= \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$ — Replace a with 1, b with -2, and c with 2.

$= \dfrac{2 \pm \sqrt{-4}}{2}$ — Simplify.

$= \dfrac{2 \pm 2i}{2}$ — $\sqrt{4} \times \sqrt{-1} = 2i$

$= 1 \pm i$ — Simplify.

Thus, the function has one real zero at $x = 2$ and two imaginary zeros at $x = 1 + i$ and $x = 1 - i$. The graph of the function verifies that there is only one real zero.



CHECK Your Progress

3. Find all of the zeros of $h(x) = x^3 + 2x^2 + 9x + 18$.

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In Chapter 5, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KEY CONCEPT — *Complex Conjugates Theorem*

Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

EXAMPLE — Use Zeros to Write a Polynomial Function

4 **Write a polynomial function of least degree with integral coefficients the zeros of which include 3 and $2 - i$.**

Explore If $2 - i$ is a zero, then $2 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x - 3$, $x - (2 - i)$, and $x - (2 + i)$ are factors of the polynomial function.

Plan Write the polynomial function as a product of its factors.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$

Solve Multiply the factors to find the polynomial function.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$	Write an equation.
$= (x - 3)[(x - 2) + i][(x - 2) - i]$	Regroup terms.
$= (x - 3)[(x - 2)^2 - i^2]$	Rewrite as the difference of two squares.
$= (x - 3)[x^2 - 4x + 4 - (-1)]$	Square $x - 2$ and replace i^2 with -1.
$= (x - 3)(x^2 - 4x + 5)$	Simplify.
$= x^3 - 4x^2 + 5x - 3x^2 + 12x - 15$	Multiply using the Distributive Property.
$= x^3 - 7x^2 + 17x - 15$	Combine like terms.

Check Since there are three zeros, the degree of the polynomial function must be three, so $f(x) = x^3 - 7x^2 + 17x - 15$ is a polynomial function of least degree with integral coefficients and zeros of 3, $2 - i$, and $2 + i$.

CHECK Your Progress

4. Write a polynomial function of least degree with integral coefficients the zeros of which include -1 and $1 + 2i$.

CHECK Your Understanding

Example 1 (p. 363)

Solve each equation. State the number and type of roots.

1. $x^2 + 4 = 0$ **2.** $x^3 + 4x^2 - 21x = 0$

Example 2 (p. 364)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

3. $f(x) = 5x^3 + 8x^2 - 4x + 3$ **4.** $r(x) = x^5 - x^3 - x + 1$

Example 3 (pp. 364–365)

Find all of the zeros of each function.

5. $p(x) = x^3 + 2x^2 - 3x + 20$ **6.** $f(x) = x^3 - 4x^2 + 6x - 4$

7. $v(x) = x^3 - 3x^2 + 4x - 12$ **8.** $f(x) = x^3 - 3x^2 + 9x + 13$

Example 4 (pp. 365–366)

9. Write a polynomial function of least degree with integral coefficients the zeros of which include 2 and $4i$.

10. Write a polynomial function of least degree with integral coefficients the zeros of which include $\frac{1}{2}$, 3, and -3.

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–16	1
17–22	3
23–32	2
33–38	4

Solve each equation. State the number and type of roots.

11. $3x + 8 = 0$ **12.** $2x^2 - 5x + 12 = 0$

13. $x^3 + 9x = 0$ **14.** $x^4 - 81 = 0$

15. $x^4 - 16 = 0$ **16.** $x^5 - 8x^3 + 16x = 0$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

17. $f(x) = x^3 - 6x^2 + 1$

18. $g(x) = 5x^3 + 8x^2 - 4x + 3$

19. $h(x) = 4x^3 - 6x^2 + 8x - 5$

20. $q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$

21. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$

22. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

Find all of the zeros of each function.

23. $g(x) = x^3 + 6x^2 + 21x + 26$

24. $h(x) = x^3 - 6x^2 + 10x - 8$

25. $f(x) = x^3 - 5x^2 - 7x + 51$

26. $f(x) = x^3 - 7x^2 + 25x - 175$

27. $g(x) = 2x^3 - x^2 + 28x + 51$

28. $q(x) = 2x^3 - 17x^2 + 90x - 41$

29. $h(x) = 4x^4 + 17x^2 + 4$

30. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40$

31. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13$

32. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156$

Write a polynomial function of least degree with integral coefficients that has the given zeros.

33. $-4, 1, 5$

34. $-2, 2, 4, 6$

35. $4i, 3, -3$

36. $2i, 3i, 1$

37. $9, 1 + 2i$

38. $6, 2 + 2i$

PROFIT **For Exercises 39–41, use the following information.**
A computer manufacturer determines that for each employee the profit for producing x computers per day is $P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x$.

39. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (*Hint:* Notice that 0, which is neither positive nor negative, is a zero of this function since $d(0) = 0$.)

40. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator.

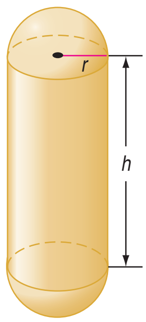
41. What is the meaning of the roots in this problem?

Real-World Link

A space shuttle is a reusable vehicle, launched like a rocket, which can put people and equipment in orbit around Earth. The first space shuttle was launched in 1981.

Source: kidsastronomy.about.com

SPACE EXPLORATION **For Exercises 42 and 43, use the following information.**

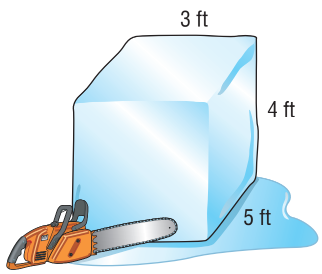


The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of 336π cubic meters and a height of 17 meters more than the radius of the tank. (*Hint:* $V(r) = \pi r^2 h$)

42. Write an equation that represents the volume of the cylinder.

43. What are the dimensions of the cylindrical part of the tank?

SCULPTING **For Exercises 44 and 45, use the following information.**



Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet.

44. Write a polynomial equation to model this situation.

45. How much should he take from each dimension?

EXTRA PRACTICE
See pages 904, 931.

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H.O.T. Problems

46. **OPEN ENDED** Sketch the graph of a polynomial function that has the indicated number and type of zeros.
 a. 3 real, 2 imaginary b. 4 real c. 2 imaginary

47. **CHALLENGE** If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation.

48. **REASONING** State the least degree a polynomial equation with real coefficients can have if it has roots at $x = 5 + i$, $x = 3 - 2i$, and a double root at $x = 0$. Explain.

49. **CHALLENGE** Find a counterexample to disprove the following statement. *The polynomial function of least degree with integral coefficients with zeros at* $x = 4$, $x = -1$, *and* $x = 3$, *is unique.*

50. ***Writing in Math*** Use the information about medication on page 362 to explain how the roots of an equation can be used in pharmacology. Include an explanation of what the roots of this equation represent and an explanation of what the roots of this equation reveal about how often a patient should take this medication.

STANDARDIZED TEST PRACTICE

51. **ACT/SAT** How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have?

 A 3

 B 2

 C 1

 D 0

52. **REVIEW** Tiles numbered from 1 to 6 are placed in a bag and are drawn out to determine which of six tasks will be assigned to six people. What is the probability that the tiles numbered 5 and 6 are drawn consecutively?

 F $\frac{2}{3}$ G $\frac{2}{5}$ H $\frac{1}{2}$ J $\frac{1}{3}$

Spiral Review

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function. (Lesson 6-7)

53. $f(x) = x^3 - 5x^2 + 16x - 7$

54. $f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5$

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

55. $15a^2b^2 - 5ab^2c^2$

56. $12p^2 - 64p + 45$

57. $4y^3 + 24y^2 + 36y$

58. **BASKETBALL** In a recent season, Monique Currie of the Duke Blue Devils scored 635 points. She made a total of 356 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 76 more 2-point field goals than free throws and 77 more free throws than 3-point field goals. Find the number of each type of shot she made. (Lesson 3-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find all values of $\pm\frac{a}{b}$ given each replacement set.**

59. $a = \{1, 5\}$; $b = \{1, 2\}$

60. $a = \{1, 2\}$; $b = \{1, 2, 7, 14\}$

61. $a = \{1, 3\}$; $b = \{1, 3, 9\}$

62. $a = \{1, 2, 4\}$; $b = \{1, 2, 4, 8, 16\}$